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Power Spectrum Parameter Estimation

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***Abstract*—The power spectrum of a zero-mean stationary Gaussian random process is assumed to be known except for one or more parameters which are to be estimated from an observation of the process during a finite time interval. The approximation is introduced that the coefficients of the Fourier series expansion of a realization of long-time duration are uncorrelated. Based on this approximation maximum likelihood estimates are derived and fundamental limits on the variances attainable are found by evaluation of the Cramér-Rao lower bound. Parameters specifically considered are amplitude, center frequency, and frequency scale factor. Also considered is ripple frequency which refers to the cosine factor in the spectrum produced by the addition of a delayed replica of the random process. The dual problem of estimating parameters of the time-varying power level of a nonstationary band-limited white noise process is examined.**

I. Introduction

THE PROBLEM of estimating parameters of the power spectrum of a stationary random process from a record of limited duration arises in a variety of applications. For example, radio astronomers have mapped the structure of our galaxy by measuring the Doppler shift of the hydrogen line in various directions in the galactic plane [1]. In radar investigations of the ionosphere, the positive ion temperature has been deduced from the spectral width of the backscattered signals [2]. Further examples pertaining to radar astronomy and seismology are described in Sections VIII and X of this paper. Some of the methods of statistical estimation theory are applied here to these problems. With the assumption that the process is Gaussian and the introduction of certain approximations valid for a long observation time, an expression is given for the likelihood function of the parameters. From this expression, maximum likelihood estimates can be determined although they cannot, in general, be stated explicitly. Fundamental limits on the variance attainable by these or any other estimates are provided by evaluation of the Cramér-Rao lower bound. Specific results are obtained for the parameters of amplitude, center frequency, frequency scale factor, and ripple frequency. It is shown that analogous results apply to the estimation of parameters governing the time variation of the power level of a nonstationary band-limited white noise process such as would be obtained from the reflection of a narrow radar pulse from a diffuse cloud of scatterers.

Many authors have examined the problem of measuring the overall shape of a power spectrum [3]–[5]. However, for situations like the above, in which the spectrum is assumed to be known except for one or more parameters, only a few treatments have appeared in the literature. Kelly, Lyons, and Root [6] and other authors analyzed

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the radiometer, which is a device for measurement of spectral amplitude. Swerling [7] and Mullen [8] described methods for estimating frequency scale factor and center frequency which require that the parameter be in the vicinity of some nominal value and that the amplitude of the power spectrum be known or stabilized by an automatic gain control. They obtained expressions for the error variances which for power spectra symmetric about a center frequency are equivalent to (37) and (41) below. Bogert, Healy, and Tukey [9] discussed a method for the measurement of the time delay of an "echo" of the random process, but presented no theoretical evaluation.

In the statistical literature, there are some specialized results dealing with parameters of moving average and autoregressive discrete time series. The only treatment which applies to a general class of parameters is the work of Whittle [10]—[12] which has not received attention proportional to its importance. He treated hypothesis testing for a discrete Gaussian time series with unknown power spectrum parameters by approximating the covariance matrix of the time series with one which is circular (periodic). The hypothesis-testing framework places the amplitude parameter in a special category and leads to the conclusion that its maximum likelihood estimate is asymptotically uncorrelated with that of the other parameters. Here (in Section IV) a different result is obtained which has been verified by other means. Otherwise, the application of our methods to discrete time series produces results consistent with Whittle's.

II. Maximum Likelihood Estimates and the Cramér-Rao Lower Bound

Some general results of statistical estimation theory [13] are presented in this section. Consider an N-dimensional vector random variable $\boldsymbol{\xi}$ with probability density $g(\mathbf{x}; \alpha)$ depending on a parameter α with true value α_0. For an observation of $\boldsymbol{\xi}$, the natural logarithm of the likelihood of α is defined as

$$\Lambda(\boldsymbol{\xi}; \alpha) = \log g(\boldsymbol{\xi}; \alpha)$$

The maximum likelihood (ML) estimate $\hat{\alpha}$ is the value of α which maximizes $\Lambda(\boldsymbol{\xi}; \alpha)$. It can sometimes be found explicitly as the root of the likelihood equation

$$\frac{\partial}{\partial \alpha} \Lambda(\boldsymbol{\xi}; \alpha) = 0 \tag{1}$$

but usually a linearization in the vicinity of α_0 or a method of successive approximations is required.

The Cramér-Rao lower bound states that under general regularity conditions for any estimate α_e

$$\operatorname{var} \alpha_e \geq \frac{[1 + db/d\alpha_0]^2}{-E\Lambda''(\boldsymbol{\xi}; \alpha_0)} \tag{2}$$

where

$$b = E\alpha_e - \alpha_0$$

is the bias of α_e. An alternate expression for the denominator of (2) is obtained from the identity

$$-E\Lambda''(\boldsymbol{\xi}; \alpha_0) = E[\Lambda'(\boldsymbol{\xi}; \alpha_0)]^2 \tag{3}$$

In this paper, a prime always denotes differentiation with respect to α and

$$\Lambda'(\boldsymbol{\xi}; \alpha_0) = \frac{\partial}{\partial \alpha} \Lambda(\boldsymbol{\xi}; \alpha)\bigg|_{\alpha = \alpha_0}$$

$$\Lambda''(\boldsymbol{\xi}; \alpha_0) = \frac{\partial^2}{\partial \alpha^2} \Lambda(\boldsymbol{\xi}; \alpha)\bigg|_{\alpha = \alpha_0}$$

An unbiased estimate whose variance satisfies (2) with the equals sign is said to be efficient. However, efficient estimates exist only in certain cases. ML estimates are not necessarily unbiased or efficient. However, ML estimates based on N independent samples of a random variable have certain optimal properties as $N \to \infty$. Grenander [14] extended these results to ML estimates based on a realization of a random process of duration T. He established under general conditions that as $T \to \infty$, $\hat{\alpha}$ converges in probability to α_0 and is asymptotically efficient. This implies that $\sqrt{-E\Lambda''(\boldsymbol{\xi}; \alpha_0)}(\hat{\alpha} - \alpha_0)$ converges in distribution to a distribution with mean zero and variance one. In addition, this limiting distribution is usually Gaussian.

Frequently the bias of an estimate under consideration cannot be established, so (2) is not informative. However, the Cramér-Rao lower bound can still be interpreted in terms of the "sensitivity" introduced by Kelly, Lyons, and Root [6]. They consider any statistic α_e which is a measure of α_0 in the sense that its expectation is a monotonic function of α_0 and define

$$\text{sensitivity} = \frac{dE\alpha_e/d\alpha_0}{\text{standard deviation of } \alpha_e \text{ for } \alpha = \alpha_0}$$

The reciprocal of the sensitivity is just that small change in α_0 required to change the mean value of α_e by one standard deviation. It is seen that (2) can now be written

$$\left(\frac{1}{\text{sensitivity}}\right)^2 \geq \frac{1}{-E\Lambda''(\boldsymbol{\xi}; \alpha_0)} \tag{4}$$

Consider now the general class of estimates α^* obtained by maximizing over α some function $\Gamma(\boldsymbol{\xi}; \alpha)$ which depends on $\boldsymbol{\xi}$ and α (but not, of course, on α_0) and has the further property

$$E\Gamma'(\boldsymbol{\xi}; \alpha_0) = 0 \tag{5}$$

The dispersion of the sampling errors can be characterized by the quantity

$$S(\alpha^*) = \frac{E[\Gamma'(\boldsymbol{\xi}; \alpha_0)]^2}{[E\Gamma''(\boldsymbol{\xi}; \alpha_0)]^2}$$

This quantity can be further interpreted when $\Gamma(\boldsymbol{\xi}; \alpha)$ is sufficiently regular so that in the vicinity of the true

parameter value α_0 it can be approximated by the first three terms of the Taylor series expansion

$$\Gamma(\xi; \alpha) \approx \Gamma(\xi; \alpha_0) + (\alpha - \alpha_0)\Gamma'(\xi; \alpha_0) + \frac{(\alpha - \alpha_0)^2}{2} \Gamma''(\xi; \alpha_0) \quad (6)$$

The maximum value of $\Gamma(\xi; \alpha)$ over α occurs where

$$\Gamma'(\xi; \alpha) = 0 \quad (7)$$

so from (6)

$$\alpha^* - \alpha_0 \approx - \frac{\Gamma'(\xi; \alpha_0)}{\Gamma''(\xi; \alpha_0)} \quad (8)$$

Then the distribution of the sampling error, $\alpha^* - \alpha_0$, is approximately the same as that of the expression (8). If the further approximation is made that $\Gamma''(\xi; \alpha_0)$ is a constant or that its random components are of a smaller order than those of the numerator of (8), then the random fluctuations of $\Gamma'(\xi; \alpha_0)$ about zero may be thought of as a linear noise term displacing the peak of the parabolic approximation to $\Gamma(\xi; \alpha)$ and

$$\operatorname{var} \alpha^* \approx S(\alpha^*) \quad (9)$$

It has been shown by Godambe [15] that if $S(\alpha^*)$ is taken as a measure of the dispersion of the estimate α^*, without necessarily referring to the particular interpretation mentioned above, then under general regularity conditions $S(\alpha^*)$ is minimized when $\Gamma(\xi; \alpha) = \Lambda(\xi; \alpha)$ and $\alpha^* = \hat{\alpha}$. This minimum value is, with (3),

$$S(\hat{\alpha}) = \frac{E[\Lambda'(\xi; \alpha_0)]^2}{[E\Lambda''(\xi; \alpha_0)]^2} = \frac{1}{- E\Lambda''(\xi; \alpha_0)} \quad (10)$$

This same quantity appears in the Cramér-Rao lower bound (2), but the present result holds with the equals sign for any regular estimate, biased or not.

III. Approximate Maximum Likelihood Estimates for Power Spectrum Parameters

The analysis is based on the following assumptions:

1) The random process is stationary, Gaussian, and zero mean with a double-sided power spectrum $P(f; \alpha)$ which is a known function of the parameter α whose value is to be estimated. (If the process consists of a random signal plus an independent noise, then $P(f; \alpha)$ is the sum of the two individual spectra.)

2) A particular realization $v(t)$ of the process is observed for $0 \leq t \leq T$.

3) The true value of α is denoted by α_0 and the actual power spectrum is $\Phi(f) = P(f; \alpha_0)$. α_0 is assumed to lie within a known finite interval $\alpha_{\min} < \alpha_0 < \alpha_{\max}$.

4) For any α within this interval, only values of $P(f; \alpha)$ within some fixed finite range $0 < f_1 < |f| < \infty$ depend on α. Since f_1 and f_2 may be anywhere outside of the range of dependence, for convenience we take $f_1 = N_1/T$ and $f_2 = N_2/T$ where N_1 and N_2 are integers. (This assumption means that $P(f; \alpha)$ has some constant shape outside of $f_1 < |f| < f_2$. The assumption $f_2 < \infty$ allows a finite-dimensional formulation of the problem and is meaningful in the usual practical situation where frequencies above some finite limit are not observable. The assumption $f_1 > 0$ is only for the convenience of eliminating zero frequency which enters unsymmetrically into the analysis.)

5) For all f and all α such that $\alpha_{\min} < \alpha < \alpha_{\max}$ the functions $P(f; \alpha)$, $\partial P(f; \alpha)/\partial f$, $\partial P(f; \alpha)/\partial \alpha$, and $\partial^2 P(f; \alpha)/\partial f \, \partial \alpha$ are bounded and continuous. Also $P(f; \alpha)$ is bounded away from zero for $f_1 - (f_2 - f_1) \leq f \leq f_2 + (f_2 - f_1)$.

In dealing with statistical inference for random processes, a basic problem is to choose a set of "observable coordinates"[1] which represent the process and whose probability distribution is tractable. In this paper, the observable coordinates are taken as the Fourier coefficients

$$\gamma_n = \frac{1}{\sqrt{T}} \int_0^T v(t) e^{-i2\pi n f_0 t} \, dt$$

where $f_0 = 1/T$. This is plausible since under general conditions, for almost every realization, $v(t)$ can be represented by the limit in the mean of its Fourier series expansion [16]. The real and imaginary parts of the γ_n have a multivariate Gaussian distribution and the likelihood function depends on the covariance matrix of the entire set of coefficients which is too complicated to be useful. We now observe [16], [17] that with the above definition

$$\lim_{T \to \infty} E\gamma_n \bar{\gamma}_m = \begin{cases} \Phi(n/T) & n = m \\ 0 & n \neq m \end{cases} \quad (11)$$

where $n \to \infty$ as $T \to \infty$ so that n/T remains constant and $\bar{\gamma}_m$ is the complex conjugate of γ_m. It can be shown with the above assumptions that this convergence goes uniformly as $\log T/T$. The approximation is now introduced that for large but finite T,

$$E\gamma_n \bar{\gamma}_m = \begin{cases} \Phi(n/T) & n = m \\ 0 & n \neq m \end{cases} \quad (12)$$

The Karhunen-Loève expansion[2] which is frequently employed in the analysis of random processes is not well adapted to spectral parameter estimation problems since it is in terms of an uncorrelated set of eigenfunctions which, except in special cases, change in a complicated way as the parameters vary. However, if and only if the covariance function of the random process is periodic with period T then the eigenfunctions become complex exponentials, the eigenvalues are values of the power spectrum spaced at intervals $1/T$, and the coefficients of the Karhunen-Loève expansion are uncorrelated [16]. Therefore, the approximation (12) is equivalent to approximating the covariance function of the process by one which is extended periodically.

The likelihood function of the parameter α is now

[1] Grenander [14], pp. 207–209.

[2] *Ibid.*, pp. 199–200; Davenport and Root, *op. cit.* [17], pp. 96–101.

obtained by substituting the observed values of the Fourier coefficients into the complex Gaussian probability density [18] which with (12), is

$$\prod_{n=N_1}^{N_2} \frac{1}{\pi P(nf_0;\alpha)} \exp\left[-\frac{|\gamma_n|^2}{P(nf_0;\alpha)}\right]$$

and depends upon $v(t)$ only through the "periodogram" $|\gamma_n|^2$. The approximate ML estimate α^* is found by maximizing the approximate log likelihood

$$\Gamma(\boldsymbol{\gamma};\alpha) = -\sum_{n=N_1}^{N_2}\left[\log \pi P(nf_0;\alpha) + \frac{|\gamma_n|^2}{P(nf_0;\alpha)}\right] \quad (13)$$

If $P(f;\alpha)$ is a slowly varying function, then the summation (13) can be approximated by an integral. The individual $|\gamma_n|^2$ need not be measured since their values can be smoothed and replaced in this integral by a continuous function $\Phi^*(f)$ which is equivalent to an approximately unbiased overall spectral estimate such as given by the lackman-Tukey method [4] or by a power spectrum analyzer. For the integral to be a good approximation, all the $|\gamma_n|^2$ must be effectively included and the resolution must be sufficient so that the detailed structure of the spectrum is not obscured. Then

$$\Gamma(\boldsymbol{\gamma};\alpha) \approx -T\int_{f_1}^{f_2}\left[\log \pi P(f;\alpha) + \frac{\Phi^*(f)}{P(f;\alpha)}\right] df \quad (14)$$

The accuracy of these estimates can be characterized in terms of the Cramér-Rao lower bound by approximating $E\Lambda''(\boldsymbol{\gamma};\alpha_0)$ with $E\Gamma''(\boldsymbol{\gamma};\alpha_0)$. We have

$$\Gamma'(\boldsymbol{\gamma};\alpha) = -\sum_n\left[\frac{P'(nf_0;\alpha)}{P(nf_0;\alpha)} - \frac{|\gamma_n|^2 P'(nf_0;\alpha)}{P^2(nf_0;\alpha)}\right] \quad (15)$$

where $P'(nf_0;\alpha) = (\partial/\partial\alpha)P(nf_0;\alpha)$. It is noted that $E\Gamma'(\boldsymbol{\gamma};\alpha_0) = 0$, satisfying (5). A second differentiation and the substitution of (3) give

$$-E\Gamma''(\boldsymbol{\gamma};\alpha_0) = \sum_n\left[\frac{P'(nf_0;\alpha_0)}{P(nf_0;\alpha_0)}\right]^2$$

$$\approx T\int_{f_1}^{f_2}\left[\frac{P'(f;\alpha_0)}{P(f;\alpha_0)}\right]^2 df \quad (16)$$

How good is this approximate analysis? Although similar approximations have been employed by Bryn [19], Good [20], Freiberger [21], and others, as well as by Whittle, a quantitative justification has not been given. We have been able to establish by somewhat involved arguments [22] (whose length discourages inclusion here) that with assumptions 1)–5) above, the approximate ML estimates are as good as the exact ML estimates in the sense that

$$\lim_{T\to\infty}\frac{S(\alpha^*)}{S(\hat{\alpha})} = 1 \quad (17)$$

It can also be shown that

$$\lim_{T\to\infty}\frac{E\Gamma''(\gamma;\alpha_0)}{E\Lambda''(\gamma;\alpha_0)} = 1 \quad (18)$$

so (16) furnishes a convenient asymptotic approximation for evaluating the Cramér-Rao lower bound. In fact (17) and (18) converge at least as rapidly as $(\log T)^4/T$. Experiences of other investigators suggest that good results are obtained for modest values of T, but a numerical guide is not yet available.

The step leading from (8) to (9) can also be justified so this, together with (10), (16), (17), and (18), enables us to state the basic conclusion that for large T

$$\operatorname{var}\alpha^* \approx \left\{\sum_{n=N_1}^{N_2}\left[\frac{P'(nf_0;\alpha_0)}{P(nf_0;\alpha_0)}\right]^2\right\}^{-1}$$

$$\approx \left\{T\int_{f_1}^{f_2}\left[\frac{P'(f;\alpha_0)}{P(f;\alpha_0)}\right]^2 df\right\}^{-1}$$

$$= \left\{T\int_{f_1}^{f_2}\left[\frac{\partial}{\partial\alpha}\log P(f;\alpha)\right]^2_{\alpha=\alpha_0} df\right\}^{-1} \quad (19)$$

and these expressions are also asymptotic values for the Cramér-Rao lower bound.

IV. Joint Estimates

For a set of M unknown parameters α_i, denoted by the vector $\boldsymbol{\alpha}$, joint ML estimates are found by maximizing $\Lambda(\boldsymbol{\xi};\alpha)$ over $\boldsymbol{\alpha}$. There is a corresponding extension of the Cramér-Rao lower bound for any set of joint unbiased estimates $\boldsymbol{\alpha}_e$. The covariance matrix, denoted by $[\operatorname{cov}(\boldsymbol{\alpha}_e)]$, has elements

$$\operatorname{cov}\alpha_{ei},\alpha_{ej} = E[\alpha_{ei} - E\alpha_{ei}][\alpha_{ej} - E\alpha_{ej}]$$

Let $[\mathfrak{L}]$ be the matrix with elements

$$\mathfrak{L}_{ij} = -E\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}\Lambda(\boldsymbol{\xi};\boldsymbol{\alpha})\bigg|_{\alpha=\alpha_0}$$

Then the matrix

$$[\operatorname{cov}(\boldsymbol{\alpha}_e)] - [\mathfrak{L}]^{-1}$$

is positive semidefinite. This implies that

$$\operatorname{var}\alpha_{ei} \geq 1/\mathfrak{L}_{ii} \quad (20)$$

for $i = 1, 2, \cdots, M$. Analogous results can be worked out for the sensitivity and the spread.

For the power spectrum parameter estimation problem we find, corresponding to (16),

$$\mathfrak{L}_{ij} = T\int_{f_1}^{f_2}\frac{\partial P(f;\alpha)}{\partial\alpha_i}\frac{\partial P(f;\alpha)}{\partial\alpha_j}\bigg|_{\alpha=\alpha_0} df \quad (21)$$

A case which is important in practical applications is that in which the amplitude level A of the spectrum is one of the unknown parameters, so that the spectrum has the form $AP(f;\alpha)$. Frequently A is a nuisance parameter whose value is not required. This often occurs in electronic equipment where the absolute value of the gain is difficult to calibrate. For a fixed value of α (see Section V),

$$A^* = \frac{1}{T(f_2 - f_1)}\sum_n\frac{|\gamma_n|^2}{P(nf_0;\alpha)}$$

Substituting this expression back into $\Gamma(A, \alpha)$ gives the log likelihood in terms of α alone. From this it can be shown that α^* can be obtained by maximizing over α

$$\Gamma(A^*, \alpha) = -\left\{\sum_n \log \pi P(nf_0; \alpha) + T(f_2 - f_1) \log\left[\sum_n \frac{|\gamma_n|^2}{P(nf_0; \alpha)}\right]\right\} + \text{constant}$$
$$\approx -T\left\{\int_{f_1}^{f_2} \log \pi P(f; \alpha)\, df + (f_2 - f_1) \log\left[T \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(f; \alpha)}\, df\right]\right\} + \text{constant} \tag{22}$$

without the necessity of determining A^*.

The effect of the lack of knowledge of A is placed in evidence by evaluating the approximation to the covariance matrix of α^* and A^*. The element corresponding to var α^* is

$$\text{var } \alpha^* \approx \left\{T \int_{f_1}^{f_2} \left[\frac{P'(f; \alpha_0)}{P(f; \alpha_0)}\right]^2 df - \frac{T}{(f_2 - f_1)} \left[\int_{f_1}^{f_2} \frac{P'(f; \alpha_0)}{P(f; \alpha_0)}\, df\right]^2\right\}^{-1} \tag{23}$$

When this is compared with (19) it is seen that the lack of knowledge of A increases var α^* unless the second integral in (23) is zero.

V. Estimation of Spectral Amplitude

The power spectrum is assumed to be

$$P(f; A) = AS(f) + N_0$$

where $S(f)$ and N_0 are known and the parameter A is to be estimated. Then from (13)

$$\Gamma(\gamma; A) = -\sum_{n=N_1}^{N_2} \left\{\log \pi[AS(nf_0) + N_0] + \frac{|\gamma_n|^2}{[AS(nf_0) + N_0]}\right\} \tag{24}$$

and

$$\Gamma'(\gamma; A) = -\sum_n \frac{S(nf_0)[AS(nf_0) + N_0 - |\gamma_n|^2]}{[AS(nf_0) + N_0]^2} \tag{25}$$

From (19)

$$\text{var } A^* \approx \left\{\sum_n \left[\frac{S(nf_0)}{A_0 S(nf_0) + N_0}\right]^2\right\}^{-1} \approx \left\{T \int_{f_1}^{f_2} \left[\frac{S(f)}{A_0 S(f) + N_0}\right]^2 df\right\}^{-1} \tag{26}$$

A general explicit solution for A^* is not known. However, there are two significant special cases. The first is for $N_0 = 0$. Then setting $\Gamma'(\gamma; A) = 0$, we obtain

$$A^* \approx \frac{1}{f_2 - f_1} \int_{f_1}^{f_2} \frac{\Phi^*(f)}{S(f)}\, df \tag{27}$$

This is an unbiased estimate and the variance is found to be exactly

$$\frac{\text{var } A^*}{A_0^2} = \frac{1}{T(f_2 - f_1)} \tag{28}$$

Therefore, as $(f_2 - f_1)$ approaches infinity the variance approaches zero for any fixed T. This result would be expected from the well-known singularity of this situation [23].

A second special case occurs when $AS(f) \ll N_0$ for all f. A^* can be approximated by setting

$$AS(nf_0) + N_0 \approx N_0$$

in the denominator of (25) so

$$\Gamma'(\gamma; A) \approx \sum_n \frac{S(nf_0)}{N_0^2}\left[AS(nf_0) + N_0 - |\gamma_n|^2\right] \tag{29}$$

Setting (29) equal to zero gives

$$A^* \approx \frac{\int_{f_1}^{f_2} [\Phi^*(f) - N_0]S(f)\, df}{\int_{f_1}^{f_2} S^2(f)\, df} \tag{30}$$

This estimate is also unbiased and

$$\frac{\text{var } A^*}{A_0^2} \approx \frac{N_0^2}{T \int_{f_1}^{f_2} [A_0 S(f)]^2\, df} \tag{31}$$

The estimates (27) and (30) are both linear functionals of $\Phi^*(f)$ and can be obtained as linear functions of the outputs of suitable radiometers. A radiometer consists of a filter with transfer function $H(f)$ followed by a square law detector and integrator. To evaluate the estimate (27) it is required that $|H(f)|^2 = 1/S(f)$ while for (30), $|H(f)|^2 = S(f)$. The variances (28) and (31) agree with known radiometer formulas.

VI. Estimation of Center Frequency

In this case, the power spectrum has the form

$$P(f; f_c) = P(f - f_c) \quad f \geq 0$$

and

$$\Gamma(\gamma; f_c) = -\sum_{n=N_1}^{N_2} \log \pi P(nf_0 - f_c) - \sum_{n=N_1}^{N_2} \frac{|\gamma_n|^2}{P(nf_0 - f_c)}$$
$$\approx -T \int_{f_1}^{f_2} \log \pi P(f - f_c)\, df - T \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(f - f_c)}\, df \tag{32}$$

Let us further assume in this section that

$$\Phi(f_1) = \Phi(f_2) = \Phi_0 \tag{33}$$

for all allowable f_c so that $\Phi(f)$ has the form indicated in Fig. 1. Then the first integral in (32) is independent of f_c and f_c^* is obtained by *minimizing* over f_c

$$\int_{f_1}^{f_2} \frac{\Phi^*(f)}{P(f - f_c)}\, df \tag{34}$$

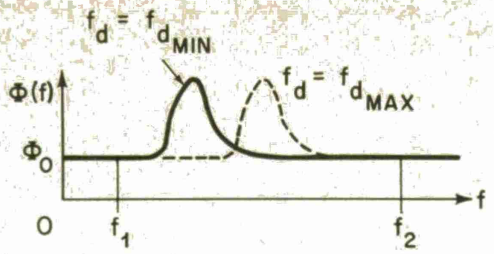

Fig. 1. The assumed spectral shape for the estimation of Doppler shift.

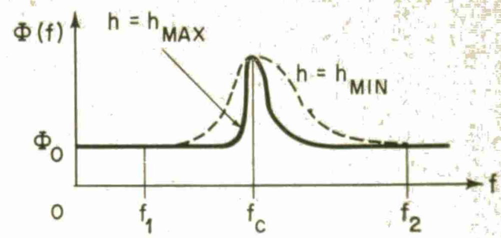

Fig. 2. The assumed spectral shape for the estimation of frequency scale factor.

This can be thought of as a "cross-correlation" between $\Phi^*(f)$ and the reciprocal of $P(f - f_c)$. Knowledge of the amplitude level of the spectrum is not required.

On the other hand if

$$P(f - f_c) = \Phi_0 + p(f - f_c) \quad f \geq 0 \tag{35}$$

where $|p(f - f_c)| \ll \Phi_0$ for all f, then f_c^* is found approximately by maximizing

$$\int_{f_1}^{f_2} \Phi^*(f)p(f - f_c)\, df \tag{36}$$

Var f_c^* is now evaluated from (19). For $f \geq 0$,

$$P'(f - f_c) = -\frac{d\Phi(f)}{df}$$

and (19) becomes

$$\text{var } f_c^* \approx \left\{ T \int_{f_1}^{f_2} \left[\frac{d\Phi(f)}{df} \Big/ \Phi(f) \right]^2 df \right\}^{-1} \tag{37}$$

The features of the spectrum which make possible the estimation of f_c are measured in a sense by the integral in (37). The integrand is non-negative at all frequencies and is large where $\Phi(f)$ is changing rapidly. From (23) it is found that when (33) holds, the lack of knowledge of the spectral amplitude does not increase the approximation to var f_c^*.

VII. Estimation of Frequency Scale Factor (Spectral Width)

The power spectrum is assumed to be

$$P(f; h) = P[h(f - f_c)] \quad f \geq 0$$

where f_c is a known center frequency and $P[h(f - f_c)]$ is known except for the frequency scale factor h which is to be estimated. (Note that the Doppler effect actually is a change of scale factor with $f_c = 0$ and the Doppler shift is an approximation valid when the width of the power spectrum is small compared with its center frequency.) h^* is obtained by maximizing

$$\Gamma(\gamma; h) \approx -T \left[\int_{f_1}^{f_2} \log \pi P[h(f - f_c)]\, df + \int_{f_1}^{f_2} \frac{\Phi^*(f)\, df}{P[h(f - f_c)]} \right] = -T[I_1 + I_2] \tag{38}$$

Now assume (33) holds for all allowable h so that $\Phi(f)$ has the form indicated in Fig. 2. Then the evaluation of I_1 can be simplified by writing

$$P[h(f - f_c)] = \Phi_0\{1 + p[h(f - f_c)]\} \tag{39}$$

By a simple transformation of variables

$$\Gamma(\gamma; h) \approx -T \left[\frac{K}{h} + \int_{f_1}^{f_2} \frac{\Phi^*(f)}{P[h(f - f_c)]}\, df \right] + \text{constant} \tag{40}$$

where

$$K = \int_{f_1}^{f_2} \log \pi [1 + p(f - f_c)]\, df$$

The maximization of (40) to determine h^* is seen to depend upon the knowledge of the spectral amplitude. If it is unknown then the method of Section IV can be used.

The Cramér-Rao lower bound (19) is found to be

$$\text{var } h^* \approx \left\{ T \int_{f_1}^{f_2} (f - f_c)^2 \left[\frac{d\Phi(f)}{df} \Big/ \Phi(f) \right]^2 df \right\}^{-1} \tag{41}$$

In comparison with (37), the integrand in (41) has an additional factor $(f - f_c)^2$ which emphasizes the spectral variations by the square of their distances from f_c. Also in contrast to the estimation of f_c, an evaluation of (23) for the present case shows that when the spectral amplitude is unknown, var h^* is, in general, increased.

VIII. Estimation of Ripple Frequency

A problem which arises in seismology and other fields [9] is that for which

$$v(t) = A[u(t) + au(t - \tau)]$$

where $u(t)$ is a zero-mean stationary Gaussian random process with known power spectrum $S(f)$, A and a are unknown amplitude factors, and τ is a time delay to be estimated. Then

$$P(f; A, a, \tau) = AS(f)[1 + 2a \cos 2\pi f\tau + a^2] \tag{42}$$

and the addition of the "echo" $a\, u(t - \tau)$ produces a cosine ripple of frequency τ in the power spectrum. It can be shown that τ^* is obtained by minimizing over τ and a

$$\Gamma(\gamma; A^*, a, \tau) = \int_{f_1}^{f_2} \frac{\Phi^*(f)}{S(f)[1 + 2a \cos 2\pi f\tau + a^2]}\, df \tag{43}$$

For small a this is equivalent to maximizing over τ

$$\int_{f_1}^{f_2} \frac{\Phi^*(f)}{S(f)} \cos 2\pi f\tau \, df$$

which amounts to "whitening" $\Phi^*(f)$ and finding the peak of the cosine transform. For $|a| \ll 1$ and $f_2 \gg 1/|\tau|$ it is found that

$$\frac{\operatorname{var} \tau^*}{T^2} \approx \left\{\frac{8\pi^2}{3} a^2 T^3 (f_2^3 - f_1^3)\right\}^{-1} \tag{44}$$

IX. A Dual Problem

A problem which is dual in a certain sense to that stated in Section III is now examined. Consider a zero-mean nonstationary Gaussian random process $y(t)$ with a time-varying covariance function

$$E[y(t)y(t+\tau)] = R(t;\beta)\delta(\tau)$$

i.e., $y(t)$ is a white noise process with a power level $R(t;\beta)$ which is a function of time and of a parameter β. Suppose $R(t;\beta)$ is known except for β which is to be estimated. The following assumptions on $R(t;\beta)$ are analogous to those of Section III regarding $P(f;\alpha)$ with time and frequency interchanged: Only a finite frequency range of the $y(t)$ process can be observed so that, in effect, it has been passed through an ideal band-pass filter with response limited to frequencies $W_1 \leq f \leq W_2$ and the estimates must be based on the resulting band-limited output process $y_B(t)$. For any value of β, only values of $R(t;\beta)$ within a finite interval $t_1 < t < t_2$ depend on β. t_1 and t_2 do not depend on β and, for convenience, we take

$$t_1 = \frac{M_1}{2(W_2 - W_1)} \quad t_2 = \frac{M_2}{2(W_2 - W_1)}$$

where M_1 and M_2 are integers. This assumption is valid if $R(t;\beta)$ has a constant shape outside of or is only observable within some fixed time interval. Finally, $R(t;\beta)$ obeys certain general regularity conditions.

Introduced the normalization

$$z(t) = y_B(t)\sqrt{W_2 - W_1}$$

On the basis of the above assumptions the band-limited process $z(t)$ can be represented by the

$$2(t_2 - t_1)(W_2 - W_1) + 1$$

samples $z(n\Delta t)$ where $\Delta t = \frac{1}{2}(W_2 - W_1)$ and $n = M_1, M_1 + 1, \cdots, M_2$. The $z(n\Delta t)$ are Gaussian and to a good approximation are independent with

$$\operatorname{var} z(n\Delta t) = R(n\Delta t;\beta) \tag{45}$$

Thus the samples $z(n\Delta t)$ are duals of the real and imaginary parts of the Fourier coefficients of Section III. The logarithm of the likelihood of the set of samples is approximately

$$\Gamma(\mathbf{z};\beta) = -\frac{1}{2}\sum_{n=M_1}^{M_2}\left[\log 2\pi R(n\Delta t;\beta) + \frac{z^2(n\Delta t)}{R(n\Delta t;\beta)}\right] \tag{46}$$

This has a form almost identical with (13) and the maximum likelihood estimation of β is analogous to the discussion of the preceding sections. The result corresponding to (19) is

$$\operatorname{var} \beta^* \approx \left\{(W_2 - W_1)\int_{t_1}^{t_2}\left[\frac{R'(t;\beta_0)}{R(t;\beta_0)}\right]^2 dt\right\}^{-1} \tag{47}$$

Thus, the wider the observable bandwidth $(W_2 - W_1)$, the smaller the variance of the estimate.

X. An Application to Radar Astronomy

An interesting example of a possible application of these results is in radar astronomy [24], [25]. When a CW signal is transmitted, the energy reflected from the moon or a planet can be considered as a Gaussian random process. The radial velocity of the body can be inferred from the Doppler shift and the rotation rate from the spectral width (frequency scale factor), so the analyses of Sections VI and VII apply. The effect of thermal noise at the receiver is included by adding its constant level to the power spectrum of the received energy.

On the other hand, Section IX applies to the case where a narrow pulse of bandwidth $(W_2 - W_1)$ is reflected from a target having a large delay spread. If the target consists of many point scatterers randomly and independently distributed with a spatial density proportional to $\sqrt{R(t;\beta)}$, then the received signal approximates a Gaussian nonstationary band-limited white noise process with time-varying power level $R(t;\beta)$. The target characteristics can be estimated as the parameters of $R(t;\beta)$, the range being analogous to the center frequency and the target depth to the frequency scale factor which were discussed in Sections VI and VII. Another parameter of significance is the rapidity with which the power level falls off with range, since it provides a measure of the surface roughness. It should be noted that the regularity assumptions may not be satisfied in some cases because of the sharp leading edge of the return signal.

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